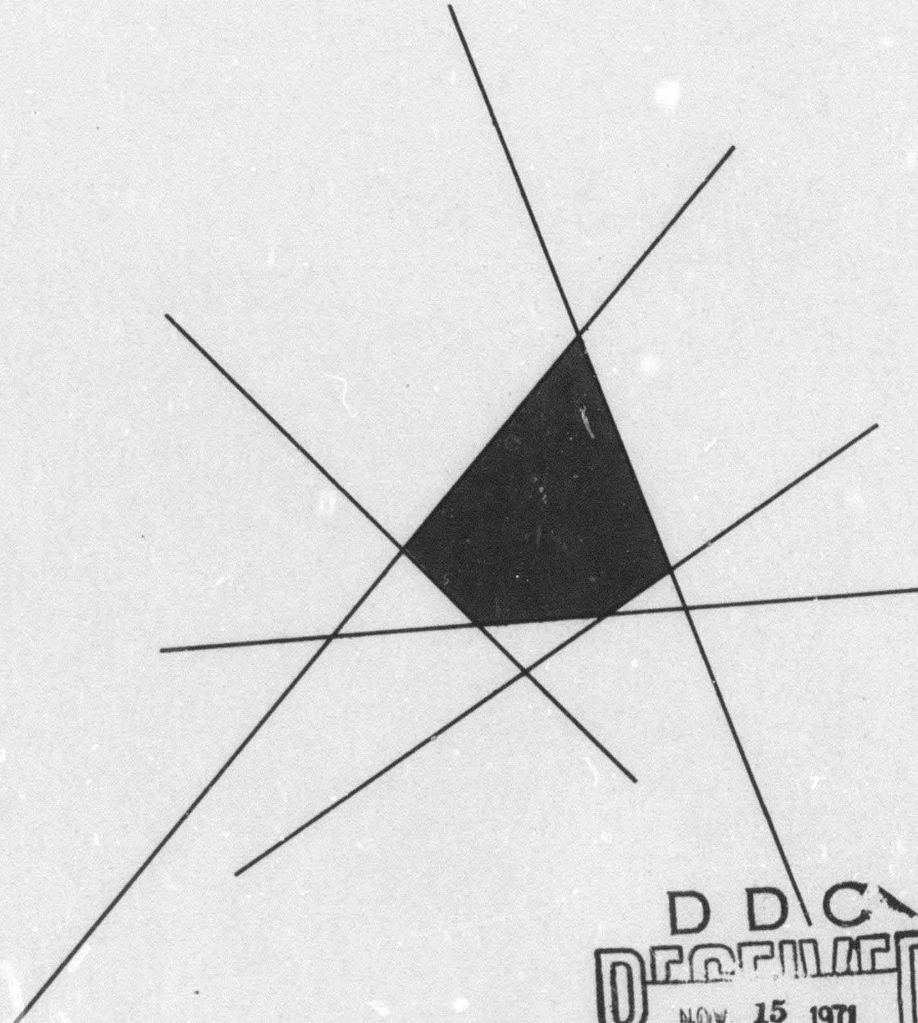
ORC 71-26
SEPTEMBER 1971

GRAPHS AND DUAL GRAPHS OF ABSTRACT POLYTOPES

by
ILAN ADLER



OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA · BERKELEY

22

GRAPHS AND DUAL GRAPHS OF ABSTRACT POLYTOPES

by

Ilan Adler
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

SEPTEMBER 1971 ORC 71-26

This research has been supported by the Office of Naval Research under Contract N00014-69-A-0200-1010 with the University of California.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

GRAPHS AND DUAL GRAPHS OF ABSTRACT POLYTOPES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5. AUTHOR(S) (First name, middle initial, last name)

Ilan Adler

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1971	18	7

8a. CONTRACT OR GRANT NO. N00014-69-A-0200-1010 b. PROJECT NO. NR 047 033 c. Research Proj. No.: RR 003 07 01 d.	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 71-26 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Mathematical Sciences Division Chief of Naval Research Department of the Navy Arlington, Virginia 22217

13. ABSTRACT

SEE ABSTRACT.

DD FORM 1 NOV 65 1473 (PAGE 1)
S/N 0101-807-6811

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Graphs						
Dual Graphs						
Polytopes						
Abstract Polytopes						

ABSTRACT

A general and convenient framework for investigating the combinatorial structure of simple polytopes is given by a set of three axioms (proposed by G. B. Dantzig) which define *abstract polytopes*. These axioms are introduced ~~in Section 1~~, and it is shown ~~there~~ that, combinatorially, simple polytopes constitute a proper subclass of abstract polytopes.

The purpose of this paper is to study the graphs and dual graphs of abstract polytopes (Sections 4 and 5, respectively) and in particular, to show that some well-known properties of graphs of polytopes ~~(given in Section 3)~~ are satisfied also by the graphs and dual graphs of abstract polytopes.

Since simple polytopes constitute (combinatorially) a subclass of abstract polytopes, the results of Section 4 show that some well-known properties of abstract polytopes, which had been previously proved using geometrical reasoning, can be proved (for simple polytopes) by applying simple combinatoric arguments.

GRAPHS AND DUAL GRAPHS OF ABSTRACT POLYTOPES

by

Ilan Adler

INTRODUCTION

A general and convenient framework for investigating the combinatorial structure of simple polytopes is given by a set of three axioms (proposed by G. B. Dantzig) which define *abstract polytopes*. These axioms are introduced in Section 1, and it is shown there that, combinatorially, simple polytopes constitute a proper subclass of abstract polytopes.

The purpose of this paper is to study the graphs and dual graphs of abstract polytopes (Sections 4 and 5, respectively) and in particular, to show that some well-known properties of graphs of polytopes (given in Section 3) are satisfied also by the graphs and dual graphs of abstract polytopes.

It should be noted that since simple polytopes constitute (combinatorially) a subclass of abstract polytopes, the results of Section 4 show that some well-known properties of abstract polytopes, which had been previously proved using geometrical reasoning, can be proved (for simple polytopes) by applying simple combinatoric arguments.

1. POLYTOPES AND ABSTRACT POLYTOPES

Given a finite set T of symbols, a family P of subsets of T (called vertices) forms a d-dimensional *abstract polytope* if the following three axioms are satisfied:

(i) Every vertex of P has cardinality d.

(ii) Any subset of $d - 1$ symbols of T is either contained in no vertices of P or in exactly two (called *neighbors* or *adjacent*).

(iii) Given any pair of vertices $v, \bar{v} \in P$, there exists a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that

(a) v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$

(b) $\{v \cap \bar{v}\} \subset v_i$ $(i = 0, \ldots, k)$.

Note that if we augment P by including all subsets of the vertices of P, then axioms (i), (ii) and (iii)(a) define a $(d - 1)$-dimensional pseudo-manifold (with no boundries).

Let us denote $UP = \{Uv \mid v \in P\}$. Let U be a subset of UP such that $|U| = k$, $0 \leq k \leq d$ ($|U|$ denotes the cardinality of U). If $P' = \{v \in P \mid U \subset v\}$ is nonempty we say that P' is the *face* of P which is *generated by* U and denote it by $F(P \mid U)$.

It is not difficult to verify that the family $\{v - U \mid v \in F(P \mid U)\}$ of subsets obtained from $F(P \mid U)$ by deleting U from each vertex of $F(P \mid U)$ is a $(d - k)$-dimensional abstract polytope. Thus, we say that $F(P \mid U)$ is a $(d - k)$-dimensional face of P. Zero, one and $(d - 1)$-dimensional faces are called, respectively, *vertices*, *edges* and *facets*. Of course, the d-dimensional face of P is P itself.

Abstract polytopes are (combinatorially) closely related to simple polytopes. A *polytope* is a bounded nonempty intersection of a finite number of closed half

spaces in a finite-dimensional real vector space. The *faces* of a polytope R are the intersections of R with its various supporting hyperplanes. Zero, one and $(d - 1)$-dimensional faces of a d-dimensional polytope R are called, respectively, the *vertices*, *edges* and *facets* of R. Two faces are said to be *incident* if one contains the other. A d-dimensional polytope is *simple* if each of its vertices is incident to exactly d edges.

Given a d-dimensional simple polytope R with n facets, assign the symbols $A_1, \ldots, A_n$ to the facets of R and define P as the family of subsets of $\{A_1, \ldots, A_n\}$ which satisfy:

> $\left\{A_{i_1}, \ldots, A_{i_d}\right\} \in P$ if, and only if the facets which correspond to $A_{i_1}, \ldots, A_{i_d}$ have a nonempty intersection.

It can be shown (see Grünbaum [5]) that P forms a d-dimensional abstract polytope which has the same face structure as R $\left(\text{i.e., } \left\{A_{i_1}, \ldots, A_{i_k}\right\}\right.$ generates a face of P if, and only if, the facets which correspond to $A_{i_1}, \ldots, A_{i_k}$ have nonempty intersection$\left.\right)$. Thus, to any d-dimensional simple polytope corresponds a d-dimensional abstract polytope with the same face structure. The converse, however, as will be shown in Section 4 by a counterexample (Figure 1) is not true.

2. NOTATION AND DEFINITIONS OF GRAPHS

A *graph* G is a pair $\{V,E\}$ where V is a (finite) set of *vertices* (or *nodes*) of G and E is a subset of the set $\{\{v_1,v_2\} \mid v_i \in V, i = 1,2; v_1 \neq v_2\}$ of pairs of members of V. The elements of E are called *edges*.

A pair of vertices are *adjacent* or *neighbors* in G if both vertices are contained in some edge of E. An edge in E is *adjacent* to a vertex v in V if it contains v.

A graph G is j-valent if every vertex of G is adjacent to exactely j edges.

If V' is a subset of vertices of G, then $G - V'$ denotes the subgraph of G obtained by omitting from G the vertices of V' and all the edges adjacent to them.

We denote by $N_G(V')$ the set of all vertices in G which are not in V' but are adjacent to at least one vertex of V'.

A *path* in G with *endpoints* $v,\bar{v}$ is a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that v_i, v_{i+1} $(i = 0, \ldots, k-1)$ are neighbors. If $v = \bar{v}$ and all other vertices are distinct, then the path is called a *simple cycle*. Two paths in G are *disjoint* if no vertex (except possibly the endpoints) is contained in both paths.

A graph G is *connected* provided that for every pair of vertices of G there exists a path in G having these vertices as endpoints. A graph G is *k-connected* provided that for every pair of vertices of G there exists k pairwise disjoint paths in G having these vertices as endpoints (obviously G is connected if, and only if, G is 1-connected).

In the next sections we shall use the following theorem which gives necessary and sufficient conditions for a d-connectedness of a graph.

<u>Theorem 2.1</u>: (Whitney [7])

A graph G with at least $k + 1$ vertices is k-connected if, and only if, every disconnecting set W of vertices of G has cardinality of at least k (W is a *disconnecting set* of G if $G - W$ is not connected).

3. GRAPHS OF POLYTOPES

The *graph* $G(R)$ *of a polytope* R is defined as the graph whose vertices and edges correspond $1 - 1$ to the vertices and edges of R. To simplify notation we shall identify the vertices and edges of R with the vertices and edges of $G(R)$.

In this section we present four well-known properties of graphs of polytopes.

Theorem 3.1: (Steinitz, see Grunbaüm [5])

A graph G is a graph of some three-dimensional polytope if, and only if, G is planar and three-connected.

Theorem 3.2:

Let R be a polytope, R' be a face of R and v, $\bar{v}$ be vertices in $R - R'$. There exists a path joining v and $\bar{v}$ in $G(R)$ such that no vertex of this path is contained in R'.

Theorem 3.2 simply states that the graph, formed by removing all the vertices of some face from the graph of a polytope, is connected. The proof is a simple exercise and is left to the reader.

Theorem 3.3: (Balinski [3])

The graph of every d-dimensional polytope is d-connected.

Theorem 3.4: (Barnett [4])

Let R be a d-dimensional polytope and let v_0 be an arbitrary vertex of R. There exists d vertices $v_1, \ldots, v_d$ in $N_{G(R)}(v_0)$ and disjoint paths p_{ij} in $G(R) - v_0(i = 1, \ldots, d - 1, j = i + 1, \ldots, d)$ such that p_{ij} has v_i, v_j as its endpoints.

4. GRAPHS OF ABSTRACT POLYTOPES

The *graph* $G(P)$ *of an abstract polytope* P is defined as the graph whose vertices and edges correspond $1 - 1$ to the vertices and edges of P. To simplify notation we shall identify the vertices and edges of P with the vertices and edges of $G(P)$.

We can now restate the third axiom of abstract polytopes using the notion of graphs. This alternative axiom will be very useful in the sequel.

Lemma 4.1: (Adler [1])

The third axiom of abstract polytope is satisfied if, and only if, the graph of every face of P is connected.

We shall show now that the Steinitz conditions (Theorem 3.1) are not necessarily satisfied by three-dimensional abstract polytopes (and hence, not to every abstract polytope corresponds a simple polytope with the same face structure).

Consider the three-dimensional abstract polytope P^* whose graph is given in Figure 1.

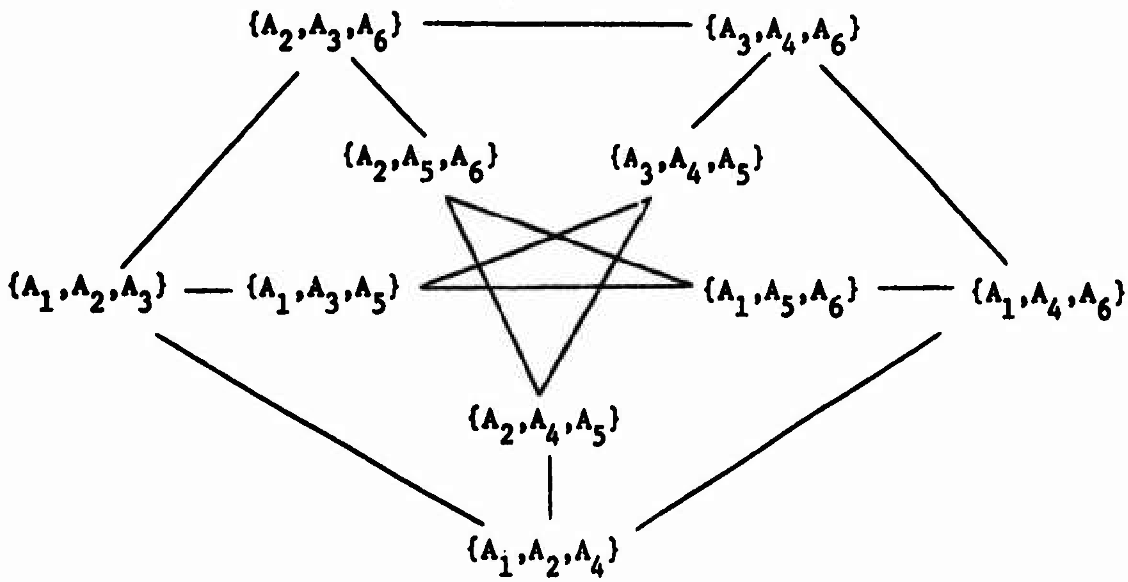


FIGURE 1

Observe that $G(P^*)$ is the well-known Peterson graph which is nonplanar.

However, there exists a set of necessary and sufficient conditions for a graph to be a graph of some three-dimensional abstract polytope. Obviously, these conditions (which are given in Theorem 4.1) are weaker than those of Steinitz.

Theorem 4.1: (Murty [6])

A graph G is a graph of some three-dimensional abstract polytope with n facets if, and only if:

1) G is a three-valent graph.

2) G is a union of n simple cycles $c_1, \ldots, c_n$ where:

   (2a) c_i , $c_j (i \neq j, i,j = 1, \ldots, n)$ have at most one common edge.

   (2b) Every edge of G belongs to exactly two cycles.

Proof:

(i) Let P be a three-dimensional abstract polytope with n facets. It is easy to verify that the graphs of the facets of P are simple cycles which satisfy (2a) and (2b) and that $G(P)$ is three-valent.

(ii) Let G be a three-valent graph which is a union of n simple cycles $c_1, \ldots, c_n$ satisfying (2a) and (2b) . Assign the symbol A_i to every vertex of $c_i (i = 1, \ldots, n)$. Thus, to every vertex of G we assign a subset of three symbols out of $\{A_1, \ldots, A_n\}$. It is easy to verify that the family of all those subsets constitutes a three-dimensional abstract polytope with n facets. ||

The last theorem gives rise to the interesting problem of facets ambiguity; namely, can a graph G be the graph of two combinatorially different three-dimensional abstract polytope? (Or even stronger, can a graph G be the graph of two three-dimensional abstract polytopes with different number of facets?) These questions are still unanswered.

So far we concentrated on the differences between simple and abstract polytopes. In the rest of the paper we shall discuss the similarities of the two, that is we shall present some properties which are satisfied by both simple and abstract polytopes. The next three theorems are the equivalent of Theorems 3.2 - 3.4 for abstract polytopes.

Theorem 4.2: (Adler, Dantzig and Murty [2])

Let P be an abstract polytope and let $A \in UP$. If $v, \bar{v} \in P - F(P \mid A)$ (i.e., if both v and $\bar{v}$ do not contain A) then there exists a path (called an *A-avoiding path*) joining v and $\bar{v}$ in $G(P)$ such that no vertex of that path belongs to $F(P \mid A)$.

The proof of the equivalence of Theorem 3.2 for abstract polytopes was first presented by Adler, Dantzig and Murty [2]. Here we give a somewhat simpler proof.

Proof:

Let P be a d-dimensional abstract polytope.

(a) $d \leq 1$: the proof is trivial.

(b) $d = 2$; by axiom (ii), $G(P)$ forms a simple cycle whose edges correspond to the facets of P. Obviously, removing an edge from $G(P)$ cannot disconnect $G(P)$.

(c) $d \geq 3$: let $P' = F(P \mid A)$ and let $v, \bar{v} \in P - P'$. By axiom (iii) there exists a sequence of adjacent vertices $v = v_0, \ldots, v_k = \bar{v}$. By axiom (ii) if $v_i \in P'$ then there exists a unique vertex $\bar{v}_i$ such that $\bar{v}_i$ is a neighbor of v_i and $\bar{v}_i \notin P'$. Let

$$u_i = \begin{cases} v_i & v_i \notin P' \\ \bar{v}_i & v_i \in P' \end{cases} \qquad (i = 1, \ldots, k) .$$

(Note that the u_i need not be distinct.) Since $|u_i \cap u_{i+1}| \geq d - 2$ $(i = 1, \ldots, k)$, $\{u_i \cap u_{i+1}\}$ generates an m dimensional face of P with $m \leq 2$, thus there exists, by (a) - (b), an A-avoiding path p_i on $F(P \mid \{u_i \cap u_{i+1}\})$ joining u_i and $u_{i+1} (i = 1, \ldots, k)$. Hence $p = \bigcup_{i=1}^{k} p_i$ is an A-avoiding path in P joining v to $\bar{v}$. ||

Corollary:

Let P be an abstract polytope and P' be a face of P then $G(P) - P'$ is connected.

Theorem 4.3:

Let P be a d-dimensional abstract polytope. The graph $G(P)$ of P is d-connected.

Proof:

For $d \leq 2$ the proof is trivial. Let $d \geq 3$ and assume the theorem is true for $d - 1$. Let V' be a minimal disconnecting set of vertices of $G(P)$ and let $v_0 \in V'$. Since V' is minimal there exists at least two vertices v_1 , v_2 in $N_{G(P)}(v_0)$ which are disconnected in $G(P) - V'$. Moreover, since $d \geq 3$, v_1 , v_2 are contained in some facet P' of P . But P' is a (d-1)-dimensional abstract polytope, hence by the induction assumption and Theorem 2.1

$$|V' \cap P'| \geq d - 1 .$$

By axiom (ii), every vertex of P' has an adjacent vertex in $P - P'$ and by the corollary to Theorem 4.2 the graph of $P - P'$ is connected. Thus, since V' is a disconnecting set of $G(P)$,

$$V' \cap \{P - P'\} \neq \emptyset .$$

From the last two expressions we obtain that

$$|V'| \geq d$$

which, by Theorem 2.1, completes the proof. ||

Theorem 4.4:

Let P be a d-dimensional abstract polytope and let v_0 be an arbitrary vertex of P and $v_1, \ldots, v_d$ its adjacent vertices. There exist disjoint paths p_{ij} in $G(P) - v_0 (i = 1, \ldots, d-1; j = i+1, \ldots, d)$ such that p_{ij} has v_i, v_j as its endpoints.

Proof:

The proof is trivial for $d \leq 2$. Let $d \geq 3$ and assume the theorem holds for $d - 1$. Let v_0 be an arbitrary vertex of P with neighbors $v_1, \ldots, v_d$. Since $v_2, \ldots, v_d$ are contained in some facet P' of P, there exist (by the induction assumption) disjoint paths $p_{ij} (2 \leq i \leq d-1, i+1 \leq j \leq d)$ in $G(P')$ such that p_{ij} has v_i, v_j as its endpoints. Since v_1, $v_j (j = 2, \ldots, d)$ are adjacent to v_0, $Q_j = F(P \mid \{v_0 \cap v_1 \cap v_j\})$ is a two-dimensional face of P. Let $p_{1j} (j = 2, \ldots, d)$ be a path from v_1 to v_j in $G(Q_j)$ which does not contain v_0. Then, p_{1j} (except the endpoint v_j) is contained in $P - P'$ and the only vertex which is contained in both p_{1j} and $p_{1k} (j,k = 2, \ldots, d; j \neq k)$ is v_1. Hence $p_{ij} (i = 1, \ldots, d-1; i+1 \leq j \leq d)$ are the required paths. ||

An attempt was made to generalize Theorem 4.1 to higher dimensions. The following conjecture was stated (and erroneously proved) by Murty [6].

Conjecture:

A graph G is a graph of some d-dimensional abstract polytope if, and only if:

1) G is a d-valent graph.

2) G is a union of n simple cycles $c_1, \ldots, c_n$ where:

(2a) c_i, $c_j (i \neq j, i,j = 1, \ldots, n)$ have at most one common edge.

(2b) Every edge of G belongs to exactly d-1 cycles.

It is rather easy to show the necessity of those conditions, by considering the graphs of all the two-dimensional faces of a given d-dimensional polytopes. However, these conditions are not sufficient. The graph displayed in Figure 2 satisfied the conditions for $d = 4$, but it is not a four-connected graph; hence by Theorem 4.3 it cannot be the graph of any four-dimensional abstract polytope.

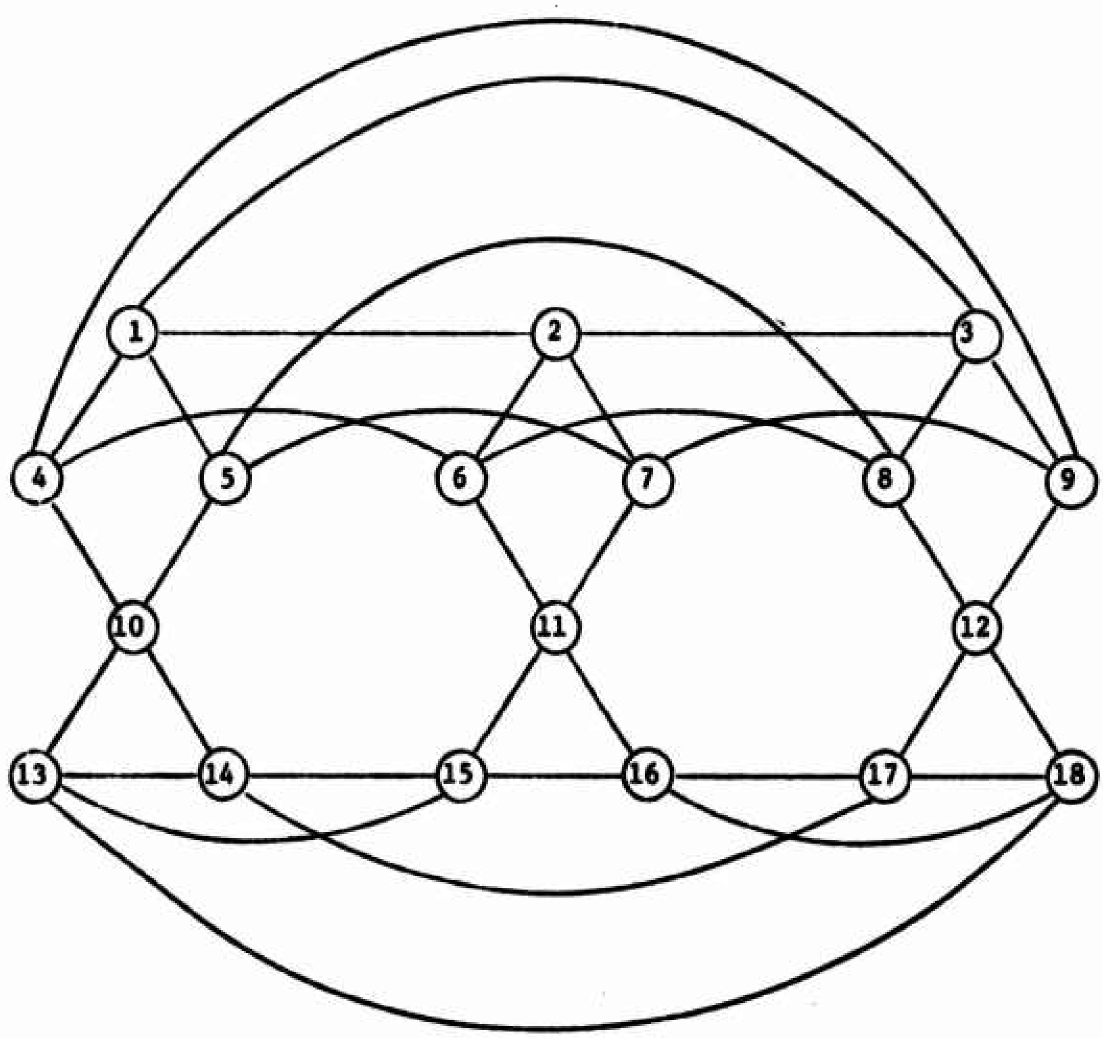


FIGURE 2

The graph in Figure 2 is the union of the following 24 cycles.

$c_1 = \{4,10,13,18,12,9,4\}$ $c_2 = \{6,11,16,18,12,8,6\}$ $c_3 = \{7,11,16,17,12,9,7\}$

$c_4 = \{5,10,14,17,12,8,5\}$ $c_5 = \{5,10,13,15,11,7,5\}$ $c_6 = \{4,10,14,15,11,6,4\}$

$c_7 = \{4,6,2,3,9,4\}$ $c_8 = \{4,1,2,7,9,4\}$ $c_9 = \{1,4,6,8,3,1\}$ $c_{10} = \{1,5,7,9,3,1\}$

$c_{11} = \{5,7,2,3,8,5\}$ $c_{12} = \{1,2,6,8,5,1\}$ $c_{13} = \{3,8,12,9,3\}$

$c_{14} = \{2,6,11,7,2\}$ $c_{15} = \{1,4,10,5,1\}$ $c_{16} = \{13,15,16,18,13\}$

$c_{17} = \{13,14,17,18,13\}$ $c_{18} = \{15,16,17,14,15\}$ $c_{19} = \{1,2,3,1\}$

$c_{20} = \{13,14,15,13\}$ $c_{21} = \{16,17,18,16\}$ $c_{22} = \{12,17,18,12\}$

$c_{23} = \{11,15,16,11\}$ $c_{24} = \{10,13,14,10\}$.

This graph is not four-connected because the removal of vertices 10, 11 and 12 disconnects the graph.

5. DUAL GRAPHS OF ABSTRACT POLYTOPES

Let P be an abstract polytope. The *dual graph* $D(P)$ of P is a graph whose set of nodes has a one to one correspondence to the set (of symbols) UP ; and an edge connects two nodes of $D(P)$ if, and only if, the pair of symbols corresponding to these nodes are contained in some vertex of P . If $v \in P$, we say that the set of nodes which corresponds to v is a *facet* of $D(P)$.

A similar definition applies to ordinary polytopes (and in particular to simple polytopes). The *dual graph* $D(R)$ of a simple polytope R is a graph whose set of nodes has a one to one correspondence to the facets of R , where an edge connects two nodes of $D(R)$ if, and only if, the two facets corresponding to these nodes have nonempty intersection.

Given any polytope R , there exists a polytope with the same dimension (called a *dual polytope*) whose graph is isomorphic to $D(R)$ (see Grünbaum [5]). Hence, all properties which are satisfied by graphs of polytopes are also satisfied by dual graphs of polytopes. In this section, we show that the same holds for abstract polytopes, i.e., that the three properties which are shown to be satisfied by graphs of abstract polytopes (Theorems 4.2-4.4) are also satisfied by dual graphs of abstract polytopes.

Theorem 5.1:

Let P be a d-dimensional abstract polytope and let v be an arbitrary vertex of P . If $A,\bar{A} \in UP - v$ then there exists a path from A to $\bar{A}$ in $D(P) - v$.

Proof:

For $d \leq 2$ the proof is trivial. Let $d \geq 3$, by Theorem 4.3 $G(P)$ is d-connected and hence there exists a path $v_0, \ldots, v_m$ in $G(P)$ such that

(a) $A \subset v_0$, $\bar{A} \subset v_m$.

(b) $v_i \neq v$ ($i = 0, \ldots, m$) .

(c) At most one of the vertices of that path is a neighbor of v .

We shall now find a path from A to $\bar{A}$ in $D(P) - v$ by applying the following procedure.

Step 1:

$i := k := 0$; $A_0 := A$ ($x := y$ means : let x be equal to y) .

Step 2:

If $A_k \not\subset v_i$ go to Step 5.

Step 3:

If $\bar{A} \subset v_i$ then $A_{k+1} := \bar{A}$; terminate.

Step 4:

$i := i + 1$; go to Step 2.

Step 5:

Find a symbol in v_{i-1} which is different from A_k and which is not contained in v and denote it by A_{k+1} ; $k := k + 1$; go to Step 2.

By axiom (ii) and the choice of $v_0, \ldots, v_m$ this procedure can be executed and it terminates after a finite number of iterations, say ℓ . Moreover, the procedure produces a sequence $A = A_0, \ldots, A_\ell = \bar{A}$ of nodes in $D(P)$ such that A_i, A_{i+1} are adjacent in $D(P)$ and $A_i \cap v = \emptyset$ ($i = 0, \ldots, \ell-1$) .||

Theorem 5.2:

The dual graph of every d-dimensional abstract polytope P is d-connected.

Proof:

The proof is similar to that of Theorem 5.1. Let v be a subset of any $d - 1$ symbols of UP (assume $d \geq 3$, since for $d \leq 2$ the proof is trivial). Given $A, \bar{A} \in UP - v$, apply the same procedure as in the proof of Theorem 5.1. The only difference is the choice of the path $v_0, \ldots, v_m$ in $G(P)$ -- replace (b) and (c) by:

(b') v_i does not contain $v (i = 0, \ldots, m)$.

Since by axiom (ii) at most two vertices of P contains v, and since, by Theorem 4.3 $G(P)$ is d-connected, there exists a path $v_0, \ldots, v_m$ in $G(P)$ which satisfies (a) and (b'). Similarly to the proof of Theorem 5.1 the procedure produces a sequence $A = A_0, \ldots, A_\ell = \bar{A}$ of nodes of $D(P)$ such that A_i, A_{i+1} are adjacent in $D(P)$ and $A_i \cap v = \emptyset$ $(i = 0, \ldots, \ell-1)$.||

Theorem 5.3:

Let P be a d-dimensional abstract polytope and let A_0 be an arbitrary symbol of UP. There exists d nodes $A_1, \ldots, A_d$ in $N_{D(P)}(A_0)$ and disjoint paths p_{ij} in $D(P) - A_0$ $(i = 1, \ldots, d-1 ; j = i+1, \ldots, d)$ such that p_{ij} has A_i, A_j as its endpoints.

Proof:

For $d \leq 2$ the proof is trivial. Suppose $d \geq 3$, let $v = \{A_0, \ldots, A_{d-1}\}$ be a vertex of P which contains A_0. By the definition of $D(P)$, A_j, A_k shares a common edge in $D(P)$ $(j \neq k; j,k = 1, \ldots, d-1)$.

Let $\bar{v}$ be a vertex of P adjacent to v such that $A_0 \subset \bar{v}$. (By axiom (ii) such vertex exists.) Without loss of generality we can assume that $\bar{v} = \{A_0, \ldots, A_{d-2}, A_d\}$. Thus A_d is adjacent to A_i $(i = 1, \ldots, d-2)$ in $D(P)$.

To complete the proof we have to show the existence of a path joining A_{d-1} and A_d in $D(P)$ which does not contain any node of $\{A_0, \ldots, A_{d-2}\}$. Theorems 5.2 and 2.1 assure the existence of such a path.||

BIBLIOGRAPHY

[1] Adler, I., " Abstract Polytopes," Ph.D. Thesis, Department of Operations Research, Stanford University, Stanford, California, (1971).

[2] Adler, I., G. B. Dantzig and K. Murty, "Existence of x-Path in Abstract Polytopes," Technical Report No. 70-1, Department of Operations Research, Stanford University, Stanford, California, (1970).

[3] Balinski, M., "On the Graph Structure of Convex Polyhedra in n-Sapce," Pacific Journal of Mathematics, Vol. 11, pp. 431-434, (1961).

[4] Barnette, D., "A Necessary Condition for d-Polyhedrality," Pacific Journal of Mathematics, Vol. 23, pp. 435-440, (1967).

[5] Grünbaum, B., CONVEX POLYTOPES, John Wiley and Sons, New York, (1967).

[6] Murty, K. G., "Necessary and Sufficient Conditions for a Regular Connected Graph to be the Graph of an Abstract Polytope," Technical Report No. 71-1, Department of Industrial Engineering, University of Michigan, Ann Arbor, Michigan, (1971).

[7] Whitney, H., "A Theorem on Graphs," Annals of Mathematics, Vol. 32, pp. 378-390, (1931).